frequency to nearly zero at 100 Hz. At 7-Hz frequency, however, the feedback increased the base vibrations by approximately 10 dB. This tradeoff between the vibration amplification and vibration attenuation over different frequency ranges was quite acceptable since at lower frequencies, the feedback in the optical pointing loop was large (and can be made even larger, if necessary, by application of nonlinear dynamic compensation,[3] and the total error in the optical loops was small.

Conclusion

The relation (6) is convenient for estimation of the effects of disturbance isolation loops in complex feedback systems.

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Terminal Area Navigation Using a Relative Global Positioning System Correction Vector Scheme

Jerome Krempasky*
Hughes Missile Systems Company, Tucson, Arizona 85734

Introduction

THE purpose of differential or relative global positioning system (GPS) schemes is to provide affordable, all-weather precision strike capability for a guided weapon without employing a terminal guidance seeker. Precision strike capability is typically understood to imply an accuracy at the target on the order of 3-m circular error probable (CEP), which is substantially better than the accuracy associated with the standard GPS precise positioning service level of 16-m spherical error probable. Differential or relative GPS schemes are currently receiving much attention because they offer the potential of meeting the U.S. Department of Defense's identification of all-weather, precision strike weaponry as a focus of modern weapon systems development. In general, the differential or relative GPS schemes proposed to date involve the use of two GPS receivers, one in the guided weapon and a second cooperative receiver, with communication between the two.[1]

The basic differential GPS scheme utilizes a cooperative GPS receiver located at a precisely known position, referred to as the base station. The purpose of the GPS receiver is to provide an estimate of the random biaslike pseudorange errors present in the pseudorange measurements for each of the visible satellites, and then to transmit these to a weapon equipped with a GPS receiver that is operating in the vicinity. The extent to which these errors are eliminated from the weapon's position solution depends on the degree of spatial and temporal correlation that these errors exhibit between the weapon and the base station. The drawback to this scheme is that the correlations degrade substantially over the times and distances typical of most tactical engagement scenarios, thus rendering the pseudorange random bias cancellation not effective. If the cooperative GPS receiver is located onboard a mobile targeting platform such as an airborne warning and control system, then the precision strike scheme is more properly referred to as relative GPS. In this implementation, the relative location of the target with respect to the targeting platform is known very accurately, e.g., through the use of the targeting aircraft's synthetic aperature radar. If the precision strike weapon's GPS receiver can maintain a high degree of correlation of its GPS position solution bias errors with those of the targeting platform, the accuracy achieved at the target can be significantly better than that of absolute GPS.[2] Note that the accuracy claims for relative GPS navigation have been verified, for separations in time and horizontal displacement, through experiments involving the placement of GPS receivers at various sites across the United States.[3]

In the relative GPS correction vector scheme considered, a high-quality terrain scene is used in place of the cooperative GPS receiver to achieve precision strike capability. The location of the terrain scene relative to the target is known very precisely, a fact that is guaranteed by the appearance of both on the same satellite photograph (geocell). At the time of overflying the terrain scene, a three-dimensional correction vector is formed in the navigation software of the precision strike weapon. This vector, which represents the difference between the estimated weapon position and the center of the terrain scene, is then applied outside of the system Kalman filter to correct all GPS positional solutions following the terrain scene. Application of the correction vector has the effect of essentially eliminating the large pseudorange bias errors inasmuch as GPS updates following the terrain scene are highly correlated with that at the terrain scene given reasonable scene/target separations. Furthermore, the weapon navigates in the terrain scene/target relative coordinate system where the absolute target errors are not important.

Correction Vector Implementation

To illustrate pictorially the performance of the relative GPS correction vector concept, the dynamics that occur in the terminal area are shown in Fig. 1. The weapon follows an actual trajectory that differs from the planned trajectory (as determined by mission planning) by the navigation position error. The primary components of the navigation position error are the pseudorange bias errors associated with the pseudorange measurements to each of the satellites being tracked. In the absence of navigation error, the planned trajectory would take the weapon to the planned world geodetic system-(WGS-) 84 coordinates of the scene, which differ from the actual scene coordinates by the large absolute scene location error. The planned WGS-84 target coordinates also differ from the actual target coordinates by the large absolute target location error, which is nearly equal to the absolute scene location error, assuming the terrain scene and target are on the same satellite photograph. Application of the correction vector at the location of the terrain scene restores the estimated trajectory (dashed line) to within a small distance of the actual scene center, i.e., the scene center error. This error arises from the inability to precisely locate the scene center, and is related to the size of the digitized terrain scene cells. The estimated trajectory will now follow a new path to the planned target. The actual trajectory will then follow a path parallel to the new path continually incorporating the correction vector. This results in the weapon arriving at the indicated impact point. This point differs from the actual target location primarily by the sum of the scene center and scene/target relative errors, which is significantly less than the sum of the pseudorange and absolute target (or scene) location random bias errors.

Application of the correction vector to GPS updates is performed external to the Kalman filter, thus resulting in no modification to the structure of the onboard filter. The corrected position solution

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*Principal Scientist, Guidance, Navigation, and Control Design, Analysis, and Simulation Department. Member AIAA.

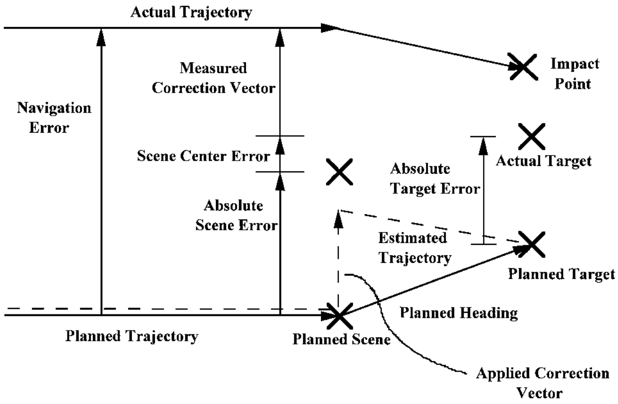

Fig. 1 Correction vector scheme pictorial.

formed by implementing the scheme is used to drive the precision strike weapon's autopilot to provide the proper guidance commands in the scene/target coordinate frame. With respect to the target, the errors incurred in this scheme can be divided into four categories; navigation, steering, scene location, and relative target location errors. The first error is the purely navigational error associated with the weapon position as estimated by the Kalman filter. This error, after subtraction of the correction vector, is driven mainly by the filtered white noise component of the GPS measurements and by weapon accelerations, both linear and angular, which excite the various instrument terms. For typical terrain scene to target distances, this error is the smallest of the four. The navigational error will, however, grow slowly with time as the GPS updates become progressively less correlated with the GPS update at the terrain scene. The second error is that associated with steering the guided weapon to the target in response to the corrected weapon position estimates and is also typically small. The third error is the accuracy inherent in the ability to locate the center of the terrain scene, whereas the final error is that of the target relative to the terrain scene. The latter two sources of error, in particular the final source of error, are the drivers governing the accuracy of the presently considered bias elimination scheme. Nonetheless, due to the small size of current digitized cells characterizing terrain scenes and the high accuracy of present day satellite imagery, the total of all four errors (in the proper rss sense) is significantly less than that achieved with GPS-only navigation.

Mathematical Prediction of Accuracy

To assess the accuracy of this approach, a method for determining the uncertainty in the position of the precision strike weapon relative to the target after application of the correction vector needs to be established. The weapon position relative to the target $\boldsymbol{x}_{tw}$ is the difference between the weapon position $\boldsymbol{x}_w$ and the target position $\boldsymbol{x}_t$, both referenced to the *ECEF* coordinate system, i.e., $\boldsymbol{x}_{tw} = \boldsymbol{x}_w - \boldsymbol{x}_t$. The weapon position, in turn, is the commanded weapon position, which is sent to the autopilot $\boldsymbol{x}_c$ plus a steering error $\delta\boldsymbol{x}_s$, which represents the ability of the weapon to steer to the commanded position. In the relative GPS correction vector scheme, the commanded weapon position at time k is the estimated weapon position as provided by the system Kalman filter, $\boldsymbol{x}_{\text{est}}(k)$, less the three-dimensional correction vector $\boldsymbol{d}$. As already discussed, the correction vector is the difference $\boldsymbol{x}_{\text{est}}(0) - \boldsymbol{x}_d$, where $\boldsymbol{x}_{\text{est}}(0)$ denotes the estimated weapon position at the time of passage over the terrain scene ($k = 0$) and $\boldsymbol{x}_d$ is the position vector locating the terrain scene relative to the *ECEF* coordinates. Taken all together, the weapon position relative to the target at time k is given by

$$\boldsymbol{x}_{tw}(k) = \boldsymbol{x}_{\text{est}}(k) - \boldsymbol{x}_{\text{est}}(0) + \delta\boldsymbol{x}_s(k) + \boldsymbol{x}_d - \boldsymbol{x}_t \tag{1}$$

The uncertainty in the weapon position is found by taking the covariance of this expression.

Because $\boldsymbol{x}_{\text{est}}(k) - \boldsymbol{x}_{\text{est}}(0)$, $\delta\boldsymbol{x}_s(k)$, and $\boldsymbol{x}_d - \boldsymbol{x}_t$ are all uncorrelated, the covariance of Eq. (1) is as follows:

$$E\left[\boldsymbol{x}_{tw}(k)\boldsymbol{x}_{tw}^T(k)\right] = E\left[\Delta\,\boldsymbol{x}_{\text{est}}(k)\Delta\,\boldsymbol{x}_{\text{est}}^T(k)\right] + E\left[\delta\boldsymbol{x}_s(k)\delta\boldsymbol{x}_s^T(k)\right] + E\left[(\boldsymbol{x}_d - \boldsymbol{x}_t)(\boldsymbol{x}_d - \boldsymbol{x}_t)^T\right] \tag{2}$$

where $\Delta\,\boldsymbol{x}_{\text{est}}(k)$ denotes $\boldsymbol{x}_{\text{est}}(k) - \boldsymbol{x}_{\text{est}}(0)$. The errors in $\boldsymbol{x}_d$ and $\boldsymbol{x}_t$ can be written, respectively, as

$$\delta\boldsymbol{x}_d = \delta\boldsymbol{x}_{ec} + \delta\boldsymbol{x}_{cd}, \qquad \delta\boldsymbol{x}_t = \delta\boldsymbol{x}_{ec} + \delta\boldsymbol{x}_{ct} \tag{3}$$

where $\delta\boldsymbol{x}_{ec}$ is the error in locating the true terrain scene center relative to *ECEF*, $\delta\boldsymbol{x}_{ct}$ is the target to terrain scene relative error, and $\delta\boldsymbol{x}_{cd}$ is the error in locating the true center of the terrain scene incurred in the process of establishing the correction vector. If the terrain scene and target are on the same satellite geocell, then $\delta\boldsymbol{x}_{ct}$ is much smaller than $\delta\boldsymbol{x}_{ec}$. Also, assuming current cell sizes for the digitized terrain scene, $\delta\boldsymbol{x}_{cd}$ is significantly less than $\delta\boldsymbol{x}_{ec}$. If it is recognized that $\delta\boldsymbol{x}_{ct}$ and $\delta\boldsymbol{x}_{cd}$ are uncorrelated, then Eq. (2) can be rewritten as

$$E\left[\boldsymbol{x}_{tw}(k)\boldsymbol{x}_{tw}^T(k)\right] = E\left[\Delta\,\boldsymbol{x}_{\text{est}}(k)\Delta\,\boldsymbol{x}_{\text{est}}^T(k)\right] + E\left[\delta\boldsymbol{x}_s(k)\delta\boldsymbol{x}_s^T(k)\right] + E\left(\delta\boldsymbol{x}_{cd}\delta\boldsymbol{x}_{cd}^T\right) + E\left(\delta\boldsymbol{x}_{ct}\delta\boldsymbol{x}_{ct}^T\right) \tag{4}$$

In this form, the four errors discussed earlier are precisely the terms appearing on the right-hand side. Moreover, they appear in the same order as discussed previously.

Whereas values for the second, third, and fourth error terms on the right-hand side of Eq. (4) can be readily assigned, a time-dependent value for the first error term, the purely navigational error, needs to be determined by means of a covariance or Monte Carlo simulation, which models the performance of the system Kalman filter. This is evident from examining the following expansion of this error term:

$$E\left[\Delta\,\boldsymbol{x}_{\text{est}}(k)\Delta\,\boldsymbol{x}_{\text{est}}^T(k)\right] = E\left[\boldsymbol{x}_{\text{est}}(k)\boldsymbol{x}_{\text{est}}^T(k)\right] - E\left[\boldsymbol{x}_{\text{est}}(k)\boldsymbol{x}_{\text{est}}^T(0)\right] - E\left[\boldsymbol{x}_{\text{est}}(0)\boldsymbol{x}_{\text{est}}^T(k)\right] + E\left[\boldsymbol{x}_{\text{est}}(0)\boldsymbol{x}_{\text{est}}^T(0)\right] \tag{5}$$

Clearly, the first and last terms here are just the position submatrices of the Kalman filter covariance matrices at time k and time 0, respectively. To evaluate the middle two terms, it is necessary to calculate the correlation that exists between $\boldsymbol{x}_{\text{est}}(k)$ and $\boldsymbol{x}_{\text{est}}(0)$ in terms of the usual Kalman filter parameters. Thus, the matrix autocorrelation function $\boldsymbol{R}(k, 0)$, which connects the Kalman filter states at the terrain scene $\boldsymbol{x}(0)$ to those at a later time $\boldsymbol{x}(k)$, must be determined.

The autocorrelation function, which is not evaluated in the normal mathematical representation of the Kalman filter, is found to be expressible in terms of the usual quantities that appear in the Kalman filter algorithm, namely, the Kalman gain $\boldsymbol{K}(k)$, the state transition matrix $\boldsymbol{\Phi}(k + 1, k)$, and the measurement matrix $\boldsymbol{H}(k)$. In particular, the matrix autocorrelation function for the first GPS update following the terrain scene is expressible as

$$\boldsymbol{R}(1, 0) = [\boldsymbol{I} - \boldsymbol{K}(1)\boldsymbol{H}(1)]\boldsymbol{\Phi}(1, 0)\boldsymbol{P}(0) \tag{6}$$

where $\boldsymbol{P}(0)$ is the error covariance matrix at the terrain scene. For later GPS updates, a computationally efficient manner of calculating the autocorrelation function has been derived, i.e.,

$$\boldsymbol{R}(k + 1, 0) = [\boldsymbol{I} - \boldsymbol{K}(k + 1)\boldsymbol{H}(k + 1)]\boldsymbol{\Phi}(k + 1, k)\boldsymbol{R}(k, 0) \tag{7}$$

Usage of this equation eliminates the need to relate all matrix autocorrelation functions back to the error covariance matrix at the terrain scene.

Covariance Simulation Predictions

Whereas it makes sense intuitively that usage of the three-dimensional correction vector should achieve high levels of accuracy because of the high correlation between the GPS update at the terrain scene and subsequent GPS updates (provided the scene/target separation is not excessive), this concept needs to be verified by a covariance simulation, where a high-quality strapdown inertial navigation system tightly coupled with a high antijam GPS receiver/digital nuller is modeled. The system Kalman filter has 21 error states, whereas the true world filter in this simulation has 87 error states. The dominant errors in GPS updating, the pseudorange random biases, appear as states only in the much larger true world filter. To predict the accuracy of the relative GPS correction vector scheme,

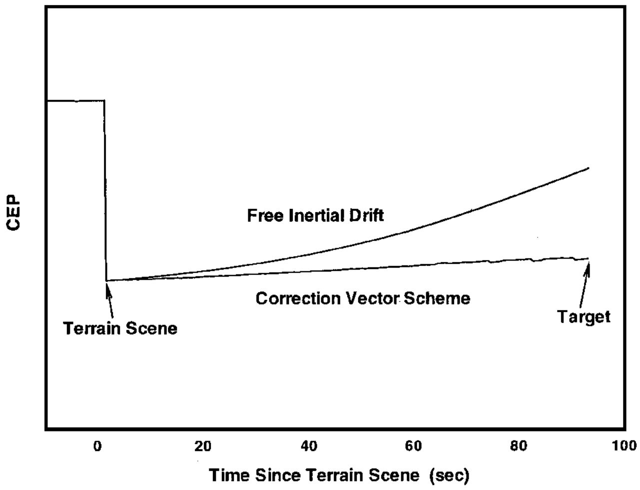

Fig. 2 Accuracy of correction vector scheme.

the covariance simulation incorporates the equations developed previously. Inasmuch as the Tomahawk missile is being considered, the terrain scene is a terminal area digital scene matching and area correlation (DSMAC) scene.

Shown in Fig. 2 is a typical CEP plot from the covariance simulation, which illustrates that the highly accurate positional solution obtained from the relative GPS correction vector scheme degrades only minimally between the DSMAC scene and the target. For target penetration considerations, the results presented here incorporate a terminal trajectory with a high-angle dive to impact the target. At the target, the navigational error contributes 2.6%, the steering error 10.2%, the scene center location error 20.9%, and the relative target location error 66.3%, which is consistent with the earlier discussion regarding relative error magnitudes. An examination of Fig. 2 also reveals that the correction vector scheme affords a substantial improvement over the situation where the Kalman filter accepts a high-quality DSMAC positional update and, subsequently, the missile flies in a free inertial manner to the target (the mode of operation for the current Block III Tomahawk missile). Because improvements in satellite imaging technology are expected to permit larger DSMAC scene to target separations while maintaining the same scene/target relative error, free inertial navigation to the target will clearly become less desirable, and relative GPS schemes such as the one proposed herein more desirable.

Summary

The advantage of employing the relative GPS correction vector scheme is basically threefold. First, through the virtual elimination of the pseudorange bias and absolute target location errors, implementation of this scheme results in accuracies at the target that are significantly better than those afforded by a standard GPS-only navigation solution. Second, the necessary modifications to existing navigation software are minimal, and thus the correction vector scheme is seen to offer affordability, especially when compared to the other primary alternative for achieving precision strike accuracies, i.e., weapons equipped with terminal guidance seekers. Continuing the comparison to terminal seekers, the proposed scheme also has an all-weather advantage because terminal seekers typically suffer degraded performance in certain environments, e.g., infrared seekers in low cloud ceilings. Finally, the proposed GPS bias elimination scheme permits significantly larger terrain scene to target separations and, hence, greater flexibility in mission planning.

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Concatenated Approach to Trajectory Optimization

Hans Seywald* and Renjith R. Kumar*
Analytical Mechanics Associates, Inc., Hampton, Virginia 23666

I. Introduction

In this Note, a new method is presented for the numerical calculation of near-optimal solutions to trajectory optimization problems. The new approach, which is applicable to a wide class of problems, relies on a scheme of sequentially solving low-order discretized subproblems on short, partly overlapping subarcs. The technique requires very little computer memory and is ideally suited for parallel processing. The overall convergence rate is probably only linear, but the calculation of individual trajectories is very fast, which makes the overall algorithm competitive also in terms of speed. Furthermore, each iterate of the trajectory satisfies all physical constraints, i.e., represents a feasible trajectory, even before complete convergence to the optimal trajectory is achieved. That means the iteration process can be stopped at any time and the currently best obtained solution can be used as a feasible trajectory.

II. Problem Formulation

Let us consider the following class of optimal control problems:

$$\min_{u \in (PWC[t_0, t_f])^m} \int_{t_0}^{t_f} L(x, u, t)\, \mathrm{d}t \tag{1}$$

subject to the conditions

$$\dot{x} = f(x, u, t) \tag{2}$$

$$\psi_0[x(t_0), t_0] = 0 \tag{3}$$

$$\psi_f[x(t_f), t_f] = 0 \tag{4}$$

Here, $x(t) : \boldsymbol{R} \to \boldsymbol{R}^n$, $u(t) : \boldsymbol{R} \to \boldsymbol{R}^m$ are the state and control functions of time, respectively, and t is the time. In addition to the conditions (2–4), we may consider control constraints and state constraints of the general form

$$g(x, u, t) \le 0 \tag{5}$$

$$h(x, t) \le 0 \tag{6}$$

respectively. The smoothness of the functions

$$L : \boldsymbol{R}^{n+m+1} \to \boldsymbol{R}, \qquad f : \boldsymbol{R}^{n+m+1} \to \boldsymbol{R}^n$$

$$\psi_0 : \boldsymbol{R}^{n+1} \to \boldsymbol{R}^{k_0}, \qquad k_0 \le n$$

$$\psi_f : \boldsymbol{R}^{n+1} \to \boldsymbol{R}^{k_f}, \qquad k_f \le n$$

$$g : \boldsymbol{R}^{n+m+1} \to \boldsymbol{R}^p, \qquad h : \boldsymbol{R}^{n+1} \to \boldsymbol{R}^q$$

with respect to their arguments is assumed to be of whatever order is required. $(PWC[t_0, t_f])^m$ denotes the set of all piecewise continuous functions defined on the interval $[t_0, t_f]$ and mapping into $\boldsymbol{R}^m$. Conditions (2–4) represent the differential equations of the

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*Staff Scientist. Member AIAA.